Physics Letters A 380 (2016) 452–457

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Understanding of flux-limited behaviors of heat transport in nonlinear regime

Yangyu Guo [a], David Jou [b], Moran Wang [a,*]

[a] Key Laboratory for Thermal Science and Power Engineering of Ministry of Education, Department of Engineering Mechanics and CNMM, Tsinghua University, Beijing 100084, China

[b] Departament de Física, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia, Spain

ARTICLE INFO

Article history:
Received 9 September 2015
Received in revised form 23 October 2015
Accepted 25 October 2015
Available online 2 November 2015
Communicated by R. Wu

Keywords:
Nanoscale heat transport
Heat flux limiter
Nonlinear heat transport
Phonon hydrodynamic model

ABSTRACT

The classical Fourier's law of heat transport breaks down in highly nonequilibrium situations as in nanoscale heat transport, where nonlinear effects become important. The present work is aimed at exploring the flux-limited behaviors based on a categorization of existing nonlinear heat transport models in terms of their theoretical foundations. Different saturation heat fluxes are obtained, whereas the same qualitative variation trend of heat flux versus exerted temperature gradient is got in diverse nonlinear models. The phonon hydrodynamic model is proposed to act as a standard to evaluate other heat flux limiters because of its more rigorous physical foundation. A deeper knowledge is thus achieved about the phenomenological generalized heat transport models. The present work provides deeper understanding and accurate modeling of nonlocal and nonlinear heat transport beyond the diffusive limit.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

Heat transport is usually described by the Fourier's law which assumes a linear dependence of heat flux on temperature gradient: $\mathbf{q} = -\lambda \nabla T$ with the coefficient λ denoting thermal conductivity of a material. This linear transport law is rigorously valid in the near-equilibrium region [1,2], where the characteristic size and time of the system are much larger than the mean free path (MFP) and relaxation time of the heat carriers respectively. In recent years, with the rapid development of micro- and nanofabrication and nanotechnology, more and more attention is focused on microscale and nanoscale heat transport [3–8] when the system size decreases to be comparable to or even smaller than the carrier MFP and the process temporal scale shortens to be close to the carrier relaxation time. The situation becomes far from equilibrium states, thus the classical Fourier's law becomes no longer available to model heat transport in this region [9].

Nanoscale heat transport includes both temporal aspects and spatial aspects. And there are usually three kinds of non-Fourier features [10]: relaxation, nonlocal, and nonlinear effects. Various generalized heat transport models involving relaxation, nonlocal and/or nonlinear terms have been proposed to tackle the issue. The first such model is the Cattaneo–Vernotte (C–V) law [11,12] which incorporates the relaxation term of heat flux into the Fourier's law to capture the temporal microscale heat transport or the so-called heat wave propagation [13,14]. The C–V law is a single-phase-lag model, which is later extended to a dual-phase-lag model [15,16] including also the relaxation term of temperature gradient. Besides the relaxation (memory) effect, several other models are developed to describe nonlocal effects in spatial microscale heat transport as well, mainly including the phonon hydrodynamic model [17] and thermon gas model [18,19], both of which have been widely applied in modeling thermal transport in nanostructures [8,19–21]. Nevertheless, the nonlinear effect, which would play an important role in nanoscale heat transport, is only taken into account in few works [22–26]. One interesting nonlinear phenomenon is the flux-limited behavior [25], where the heat flux will not increase infinitely with the temperature gradient. Instead, there exists an upper bound for the heat flux in the limit of infinite temperature gradient. This case may be met in nanostructures where a finite temperature difference (for instance, ~1 K) is established over an extremely small-scale length (for instance, ~100 nm). This kind of nonlinear behavior may have important effect, for instance, on the effective removal of heat generated in microelectronics, which is a current hot topic and big challenge [27,28]. Therefore it is of practical significance to investigate the flux-limited behavior in nanoscale heat transport.

The flux-limited behaviors were studied earlier in radiation hydrodynamics [29,30], mass diffusion [31–33], and general transport phenomena [34,35]. The existence of an upper bound of the flux

* Corresponding author. Tel.: +86 10 627 87498.
E-mail addresses: yangyuhguo@gmail.com (Y. Guo), david.jou@uab.es (D. Jou), mrwang@tsinghua.edu (M. Wang).

http://dx.doi.org/10.1016/j.physleta.2015.10.059
0375-9601/© 2015 Elsevier B.V. All rights reserved.

of transported quantity (radiative energy, mass, etc.) is attributed to the limited velocity of the carriers (for instance, the speed of light for photons). Thus the maximum flux must be smaller than the product of the limited velocity and the volumetric density of the transported quantity. There were also few works [36,37] on heat flux limiters based on transport laws in relativistic gas obtained from information theory. In recent years, the flux-limited behaviors are explored in nanostructures (e.g. silicon nanolayer in Ref. [38]), with heat flux saturation phenomenon obtained. In spite of these works, there still lacks a systematical investigation and comparison of flux-limited behaviors in different kinds of nonlinear heat transport models, especially in recent generalized laws in nanoscale heat transport. On the other hand, previous work in this field is more often phenomenological and a solid physical basis for the flux-limited behavior remains to be clarified.

The aim and organization of the present work is as below. In Section 2, we give an overview of previous nonlinear heat transport models and summarize them into three categories in terms of their theoretical foundations: phonon hydrodynamic model, nonequilibrium thermodynamic models, and phenomenological models. In Section 3, the flux-limited behaviors of nonlinear heat transport models are explored and carefully compared. In Section 4, further discussions are given on a credible physical basis and standard based on phonon kinetic theory for all the heat flux limiters. Finally, concluding remarks are made in Section 5.

2. Nonlinear heat transport models

In this section, an overview of existing nonlinear heat transport models is given. For further study and discussions of flux-limited behaviors, the nonlinear models are categorized into three types based on their theoretical origin: phonon hydrodynamic model from phonon kinetic theory, nonequilibrium thermodynamic models from irreversible thermodynamic theory, and phenomenological models from intuitive or mathematical perspective. They will be introduced respectively as below.

2.1. Phonon hydrodynamic model

Phonon systems can be described at three different levels: microscopic, mesoscopic, and macroscopic ones, with different governing equations respectively: Schrödinger's equation, phonon Boltzmann equation, and phonon hydrodynamic equations [8]. Thus phonon hydrodynamics denotes a macroscopic statistical description of phonon transport, and could be derived from a solution of phonon Boltzmann equation in phonon kinetic theory. Here the zeroth-order solution to phonon Boltzmann equation under Callaway's relaxation approximation by maximum entropy principle is considered [39,40]:

$$\tau_{\mathrm{R}}\frac{\partial\mathbf{q}}{\partial t}+\mathbf{q}=-\lambda\nabla T-\tau_{\mathrm{R}}\nabla\cdot\frac{3v_{\mathrm{g}}\langle\mathbf{qq}\rangle}{2v_{\mathrm{g}}E+\sqrt{4v_{\mathrm{g}}^{2}E^{2}-3q^{2}}},\tag{1}$$

with τ_{R} the relaxation time of phonon resistive scattering, λ the thermal conductivity of bulk material, v_{g} the average phonon group speed and E the phonon energy density. $\langle\rangle$ denotes the deviatoric part of tensor $\mathbf{qq}$. The order of solution represents different degrees of approximation around the equilibrium distribution (displaced Planck distribution) [8]. Eq. (1) is a highly nonlinear heat transport equation, in which the heat flux could be very large, i.e. far from equilibrium state [39]. Eq. (1) can be also derived from phonon Boltzmann equation by Grad's type moment method [41] and Chapman–Enskog method [42], as is thoroughly summarized in Ref. [8]. Note that the phonon energy density E in Eq. (1) is integrated from the phonon distribution function f as: $E=\int\hbar\omega f d\mathbf{k}$, with $\hbar\omega$ and $\mathbf{k}$ being respectively the energy quanta and wave vector of phonons [8,39]. The thermodynamic temperature T is defined from $E=C_V T$, which is assumed still valid in nonequilibrium situation, with C_V the heat capacity per unit volume of bulk material [4]. The thermal conductivity is derived as $\lambda=\frac{1}{3}C_V v_{\mathrm{g}}^2\tau_{\mathrm{R}}$, being exactly the value of bulk material. In the nonequilibrium situation analyzed throughout the present work, the energy density E and thermodynamic temperature T may depend on the heat flux, as is already well discussed and formulated in Refs. [1,8], whereas the heat capacity C_V and thermal conductivity λ in these nonlinear heat transport models adopt the values of bulk materials.

2.2. Nonequilibrium thermodynamic models

Thermodynamics has a close relation to heat transport model, since Fourier's law was derived in classical irreversible thermodynamics (CIT) [43]. Recent development of generalized laws in nanoscale heat transport has fostered further progress of irreversible thermodynamic theories and branches [5], one of which is known as extended irreversible thermodynamics (EIT) [1,44]. Here three nonlinear heat transport models obtained in the frame of or in the spirit of EIT will be introduced.

The first one is termed as Lagrange multiplier model, which is obtained by an identification of the coefficients in the Gibbs relation for an information-theoretical description of nonequilibrium steady state through a comparison to the generalized Gibbs relation in EIT, where the heat flux is elevated as an additional independent state variable. The heat flux is related to the Lagrange multiplier conjugate to the heat flux in information theory, which results in a nonlinear heat transport equation [37]:

$$\mathbf{q}=-\frac{1}{2}\left[1-\frac{3}{2}\left(\frac{\mathbf{q}}{v_{\mathrm{g}}C_V T}\right)^2+\sqrt{1-\frac{3}{4}\left(\frac{\mathbf{q}}{v_{\mathrm{g}}C_V T}\right)^2}\right]\lambda\nabla T.\tag{2}$$

The second one is termed as hierarchy moment model, which is developed by incorporating an infinite hierarchy of moments (energy density, heat flux, flux of heat flux, etc.) of phonon distribution function into the state variable space in the frame of EIT. It is a generalization of the hierarchy model already proposed in Ref. [1], and takes the additional effect of external force field (temperature gradient, electrical field, etc.) into the constitutive equations. Based on a continued-fraction technique, it finally gives rise to a nonlinear heat transport law as [35]:

$$\mathbf{q}=-\frac{\lambda\nabla T}{\frac{1}{2}+\sqrt{\frac{1}{4}+l^2\left(\nabla\ln T\right)^2}},\tag{3}$$

where l is the MFP of heat carriers.

The third one is termed as nonlinear phonon hydrodynamic model derived by a dynamical nonequilibrium temperature method [22,23] that could be treated as a derivative of EIT. It is not treated as phonon hydrodynamic model in Section 2.1 since it was got from mathematical formulations of thermodynamics, rather than rooted in phonon kinetic theory. The combination of the evolution equation of a semi-empirical dynamical temperature and an extended Fourier' law leads to a nonlocal and nonlinear heat transport equation [23]:

$$\tau_{\mathrm{R}}\frac{\partial\mathbf{q}}{\partial t}+\mathbf{q}=-\lambda\nabla T+\frac{2}{T}\frac{\tau_{\mathrm{R}}}{C_V}\mathbf{q}\cdot\nabla\mathbf{q}+l^2\left[\nabla^2\mathbf{q}+2\nabla(\nabla\cdot\mathbf{q})\right].\tag{4}$$

2.3. Phenomenological models

Phenomenological models are usually derived by incorporating additional nonlinear terms into the classical Fourier's law in an intuitive mathematical expression. It has sometimes a qualitative interpretation but often lacks a rigorous quantitative physical foundation. One typical such model is a tempered diffusion equation as a balance between "driven force" and "resistance force" in heat transport process [34]:

$$\alpha \frac{\nabla E}{E} + \frac{\mathbf{v}}{\sqrt{1-\mathbf{v}^2/v_g^2}} = 0. \tag{5}$$

In Eq. (5), α is the thermal diffusivity of a material, and $\mathbf{v}$ is the phonon drift velocity with $\mathbf{v} = \mathbf{q}/E$. At small temperature gradients, thus at small heat flux, Eq. (5) just reduces to the linear Fourier's law. With the increase of temperature gradient and heat flux, the drift velocity increases as well. Since the drift velocity cannot increase to an infinite value, a nonlinear term $\sqrt{1-\mathbf{v}^2/v_g^2}$ is included to put an upper bound v_g for it, inspired from the relativistic dynamics. Reformulating Eq. (5) as a heat flux form, a nonlinear heat transport equation is achieved:

$$\mathbf{q} = -\sqrt{1-\left(\mathbf{q}/v_g C_V T\right)^2}\lambda \nabla T. \tag{6}$$

Another phenomenological approach holding the similar idea of force balance is the thermon gas model proposed in recent years. The concept of thermomass is introduced as the equivalent mass of thermal energy based on Einstein's mass-energy relation. Thus the heat transport process is treated as a thermon gas flow [18,45]. The fluid mechanic equations are assumed to describe the dynamics of thermon gas, and a nonlinear heat transport equation is thus derived [19]:

$$\tau_T \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} + \lambda \nabla T = -\tau_T \nabla \cdot \left(\frac{\mathbf{qq}}{E}\right), \tag{7}$$

with τ_T the relaxation time of thermon gas and $\tau_T = \rho \tau_R v_g^2/6\gamma C_V T$, γ being the Grüneisen constant of a dielectric material.

Recently, in order to explore potential nonlinear effects of heat transport in nanostructures (nanotube, nanowires and nano thin layers), a generalized nonlinear heat transport equation is written as a combination of several previous non-Fourier models [25]:

$$\tau_R \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\lambda \left(1+\beta \mathbf{q}^2\right) \nabla T + \mu \mathbf{q} \cdot \nabla \mathbf{q} + \mu' \nabla \mathbf{q} \cdot \mathbf{q} + l^2 \left[\nabla^2 \mathbf{q} + 2\nabla(\nabla \cdot \mathbf{q})\right], \tag{8}$$

where β, μ and μ' are phenomenological coefficients, which have to be determined by comparing Eq. (8) to existing heat transport equations. A generalized heat transport equation similar to Eq. (8) is also given in Ref. [26], where an extended EIT framework is established for such nonlinear models.

3. Flux-limited behaviors

In this section, a systematical investigation is made of the flux-limited behaviors in the nonlinear heat transport models summarized in Section 2. For mathematical simplicity and clear physical illustration, the one-dimensional (1D) steady-state heat conduction under a temperature gradient dT/dx is considered. In this case, the energy balance equation of heat transport reduces to:

$$\frac{\partial q_x}{\partial x} = 0. \tag{9}$$

Eq. (1) in the phonon hydrodynamic model of Subsection 2.1 reduces to:

$$q_x = -\left(5 - \frac{4}{\sqrt{1-M^2}}\right)\lambda \frac{dT}{dx}, \tag{10}$$

where the dimensionless parameter denotes fully $M = \sqrt{3}q_x/2v_g C_V T$. As the deviatoric tensor becomes $\langle \mathbf{qq} \rangle_{xx} = 2q_x^2/3$ in 1D situation, the second term on the rightside of Eq. (1) reduces to when combined with Eq. (9):

$$-\tau_R \frac{\partial}{\partial x}\left[\frac{2v_g q_x^2}{2v_g C_V T + \sqrt{4\left(v_g C_V T\right)^2 - 3q_x^2}}\right] = -4\lambda \frac{dT}{dx}\left[1 - \frac{1}{\sqrt{1-3q_x^2/4\left(v_g C_V T\right)^2}}\right]. \tag{11}$$

Substitution of Eq. (11) into Eq. (1) exactly gives rise to Eq. (10).

Eqs. (2)–(4) in the nonequilibrium thermodynamic models of Subsection 2.2 reduce respectively to the following equations:

$$q_x = -\frac{1}{2}\left[1 - \frac{3}{2}\left(\frac{q_x}{v_g C_V T}\right)^2 + \sqrt{1 - \frac{3}{4}\left(\frac{q_x}{v_g C_V T}\right)^2}\right]\lambda \frac{dT}{dx}, \tag{12}$$

$$q_x = -\frac{\lambda}{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{l^2}{T^2}\left(\frac{dT}{dx}\right)^2}}\frac{dT}{dx}, \tag{13}$$

$$q_x = -\lambda \frac{dT}{dx}. \tag{14}$$

Eqs. (6)–(8) in the phenomenological models of Subsection 2.3 reduce respectively to the following equations:

$$q_x = -\sqrt{1-\left(q_x/v_g C_V T\right)^2}\lambda \frac{dT}{dx}, \tag{15}$$

$$q_x = -\left(1 - \frac{\rho q_x^2}{2\gamma C_V^3 T^3}\right)\lambda \frac{dT}{dx}, \tag{16}$$

$$q_x = -\left(1+\beta q_x^2\right)\lambda \frac{dT}{dx}. \tag{17}$$

In Eq. (17), one has $\beta < 0$ since otherwise it will result in non-physical infinite heat flux by increasing the temperature gradient [25].

In the limit of small heat flux (small temperature gradient), Eq. (10), Eqs. (12)–(17) reduces to the Fourier's law $q_x = -\lambda dT/dx$, as the nonlinear terms on the right side of them are negligible. In contrast, in the limit of large heat flux (large temperature gradient), the nonlinear terms will be important and can no longer be neglected. Furthermore, there exists an upper bound for the heat flux in all the nonlinear heat transport equations except in Eq. (14) corresponding to the nonlinear phonon hydrodynamic model Eq. (4). This saturation heat flux is obtained by computing the mathematical limit of q_x at infinite value of dT/dx, which is trivial in Eq. (13); in Eq. (10), Eq. (12), and Eqs. (15)–(17), the mathematical limit is achieved by making the heat flux-dependent coefficients before the bulk thermal conductivity vanishing, since only in this way could a finite heat flux be obtained as a product of a vanishing term and an infinite large term. Different saturation heat fluxes are obtained in these nonlinear heat transport models, which are summarized thoroughly in Table 1. Note that in deriving the saturation heat flux of hierarchy moment model, the kinetic expression of phonon thermal conductivity $\lambda = \frac{1}{3}C_V v_g l$ is used.

For an intuitive understanding, a practical example of 1D steady-state heat conduction in silicon sample around 300 K is provided. The thermophysical properties of silicon at 300 K are listed in Table 2, and constant properties are assumed in the present work. The flux-limited behaviors determined by Eq. (10), Eq. (12), Eq. (13), Eq. (15) and Eq. (16) in the nonlinear heat transport models are calculated and compared in Fig. 1. These nonlinear algebraic equations of heat flux are solved by iteration

Table 1
An overview and comparison of saturation heat fluxes in nonlinear heat transport models.

Theoretical foundation	Heat transport models	References	Saturation heat flux
Phonon kinetic theory	Phonon hydrodynamic model Eq. (1)	[39]	$q_{\mathrm{s,PH}} = \frac{2\sqrt{3}}{5} v_{\mathrm{g}} C_V T$
Nonequilibrium thermodynamics	Lagrange multiplier model Eq. (2)	[37]	$q_{\mathrm{s,LM}} = v_{\mathrm{g}} C_V T$
	Hierarchy moment model Eq. (3)	[35]	$q_{\mathrm{s,HM}} = \frac{1}{3} v_{\mathrm{g}} C_V T$
	Nonlinear phonon hydrodynamic model Eq. (4)	[22,23]	–
Phenomenological	Tempered diffusion model Eq. (6)	[34]	$q_{\mathrm{s,TD}} = v_{\mathrm{g}} C_V T$
	Thermon gas model Eq. (7)	[18,45]	$q_{\mathrm{s,T}} = \sqrt{2\gamma C_V T/\rho} C_V T$
	Generalized nonlinear model Eq. (8)	[25]	$q_{\mathrm{s,GN}} = 1/\sqrt{\lvert\beta\rvert}$

Table 2
Thermophysical properties of bulk silicon at 300 K.

Property	λ	ρ	C_V	γ	v_{g}
Unit	W/(m K)	kg/m^3	J/(m^3 K)	–	m/s
Value	148	2330	1.66×10^6	1.5	6400

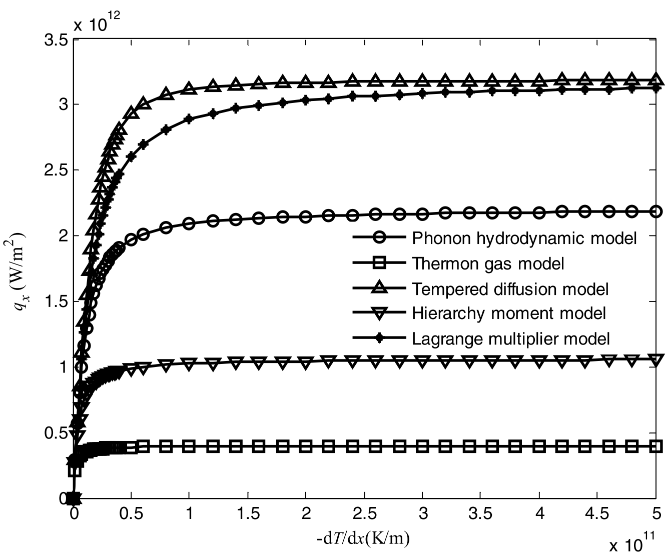

Fig. 1. A comparison of flux-limited behaviors in different nonlinear heat transport models (cf. Table 1): the heat flux versus the temperature gradient.

method and found to have one positive root or two roots (a positive one and a negative one) under a given temperature gradient. Because negative heat flux under exerted negative temperature gradient contradicts the second law of thermodynamics and is a non-physical result, the positive roots are merely kept and plotted. The parameter β in Eq. (17) has to be specified by comparing to other models (for instance, the thermon gas model), therefore the generalized nonlinear model is not included in Fig. 1. Further discussions about the flux-limited behaviors in generalized nonlinear models will be given in the following section.

The results in Fig. 1 show that heat flux increases linearly with temperature gradient at small temperature gradients, i.e. in the near-equilibrium region. At larger temperature gradients, in the far-from-equilibrium region, a nonlinear dependence of heat flux on temperature gradient is obtained. Finally, the heat flux approximates asymptotically a constant maximum value with increasing temperature gradient. The qualitative tendency of heat flux versus temperature gradient is the same, but the saturation heat flux is different in different nonlinear heat transport models. The saturation heat flux obtained in thermon gas model is about one order of magnitude smaller than those obtained in other nonlinear models. The Lagrange multiplier model has an identical saturation heat flux to that in the tempered diffusion model, but in the latter model heat flux approaches faster to the saturation value.

4. Discussions

As is seen in Section 3, diverse flux-limited behaviors are obtained in different nonlinear heat transport models. Thus it is essential to clarify the physical standard to evaluate the heat flux limiters, which is the main objective of the present section.

From our perspective, the phonon hydrodynamic model derived from phonon kinetic theory could act as a credible physical standard, since it is the most natural and direct result from phonon Boltzmann equation as the fundamental general transport law [8]. In contrast, phenomenological models are often lacking a rigorous physical foundation, because they are usually obtained by adding mathematical terms based on intuitive qualitative interpretations, such as the ‘force balance’ in tempered diffusion model; the nonequilibrium thermodynamic models cannot be an alternative as well, since the irreversible thermodynamic theory is hardly capable of producing novel transport equations although it provides a beautiful theoretical frame for the existing constitutive relations of transport process.

Therefore, the flux-limited behavior in phonon hydrodynamic model is separately compared with those in nonequilibrium thermodynamic models and those in phenomenological models, as is shown in Fig. 2(a) and Fig. 2(b) respectively. The nonequilibrium thermodynamic models have comparable flux-limited behaviors to the phenomenological models, both of which deviate from the phonon hydrodynamic model. The difference provides a credible standard to evaluate the phenomenological models or irreversible thermodynamic theories through a comparison to phonon hydrodynamic model. For instance, the thermon gas model holds a saturation heat flux (3.9877×10^{11} W/m^2) almost one order of magnitude smaller than that (2.2082×10^{12} W/m^2) in phonon hydrodynamic model; in terms of the relation $\tau_{\mathrm{T}} = \rho \tau_{\mathrm{R}} v_{\mathrm{g}}^2 / 6\gamma C_V T$ between relaxation times of thermon gas and phonon gas, the saturation heat flux in thermon gas model is rewritten as $q_{\mathrm{s,T}} = \frac{\sqrt{3}}{3}\sqrt{\frac{\tau_{\mathrm{R}}}{\tau_{\mathrm{T}}}} v_{\mathrm{g}} C_V T$. The ratio of saturation heat fluxes in both models are thus correlated to the ratio of relaxation times: $\frac{q_{\mathrm{s,T}}}{q_{\mathrm{s,PH}}} = \frac{5}{6}\sqrt{\frac{\tau_{\mathrm{R}}}{\tau_{\mathrm{T}}}}$. Actually there is usually a ratio between the quantities in phonon hydrodynamic and thermon gas models, as is carefully compared in Ref. [8], which indicates some fundamental distinction between the concept of “thermomass” and “phonon”.

The validity should be emphasized of Eq. (1), which is the hitherto available nonlinear heat transport equation in phonon hydrodynamics obtained from phonon Boltzmann equation. It is originally derived in low temperature situation, and in prior extended to describe phonon transport in a wider scope [8]. Recent works [46,47] report hydrodynamic phonon transport taking place in two-dimensional nanomaterials (graphene, etc.) even at ambient temperature based on *ab initio* calculations. Thus a fully available phonon hydrodynamic model is disperately needed. A novel concept of generalized phonon hydrodynamics has been proposed in a recent comprehensive article [8], and provides a possible avenue to describe the nonlocal and nonlinear heat transport in any range of temperature. Future exploration may be focused on the

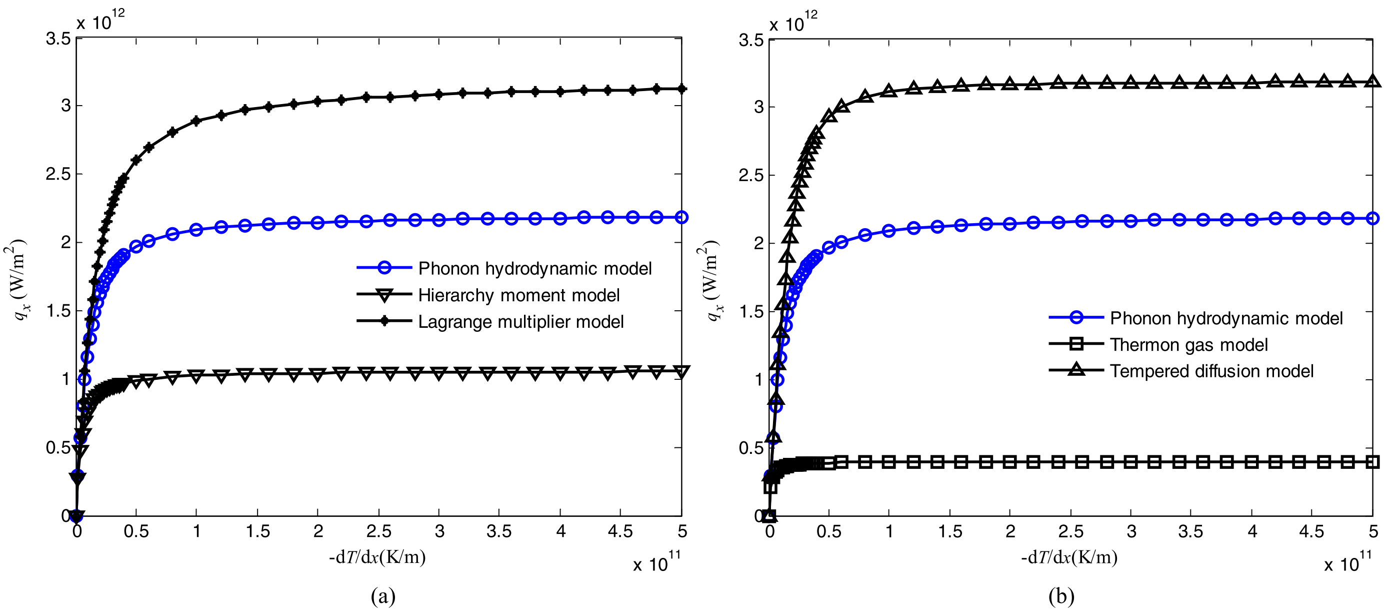

Fig. 2. Comparison of flux-limited behaviors in phonon hydrodynamic models respectively to those in the nonequilibrium thermodynamic models (a) and those in phenomenological models (b): the heat flux versus the temperature gradient.

flux-limited behaviors in the generalized phonon hydrodynamic models once they are well developed. On the other hand, Eq. (1) is the result of zeroth-order solution to phonon Boltzmann equation; higher-order solution may give rise to more refined result. However, because of the complicacy of displaced Planck distribution, first-order and higher-order solutions provide no explicit heat transport equations and lack a clear physical interpretation [8]. One may make use of particle-based numerical method such as Monte Carlo scheme [48] for solving phonon Boltzmann equation to evaluate the heat flux limiters more precisely in future work.

Finally, the flux-limited behavior obtained in Eq. (8) yields further indication about the effects of nonlinear and nonlocal terms in generalized heat transport equations. In Eq. (8), the nonlocal terms include both linear ones ($\nabla^2 \mathbf{q}$, $\nabla(\nabla \cdot \mathbf{q})$) and nonlinear ones ($\mathbf{q} \cdot \nabla \mathbf{q}$, $\nabla \mathbf{q} \cdot \mathbf{q}$), but all of them vanish in 1D steady-state conduction since the gradient term of heat flux becomes zero in steady state (Eq. (9)). Thus the nonlocal terms in Eq. (8) contribute nothing to the flux-limited behavior, which, instead, comes from the purely nonlinear term ($\mathbf{q}^2 \nabla T$) [26]. The combination of purely nonlinear term with the Fourier's term ($\lambda \nabla T$) results in a heat flux-dependent effective thermal conductivity, which is obtained in Eq. (10), Eq. (12), Eq. (15) and Eq. (16) of other nonlinear models as well. The present discussion gives also an explicit interpretation why Eq. (4) in nonequilibrium thermodynamic model induces no flux-limited behaviors.

5. Conclusions

A systematical investigation is made of the flux-limited behaviors in nonlinear regime based on a classification of existing nonlinear heat transport models into three categories: phonon hydrodynamic model, nonequilibrium thermodynamic models, and phenomenological models. The same qualitative tendency of heat flux versus exerted temperature gradient is obtained, but different values of saturation heat flux are achieved in different models. The phonon hydrodynamic model developed in phonon kinetic theory has a more rigorous physical foundation, therefore could act as a standard to evaluate other heat flux limiters. The thermon gas model is thus found to have a saturation heat flux about one order of magnitude smaller than that in phonon hydrodynamic model, which infers possible conceptual difference between thermon gas and phonon gas. It is also shown that the nonlocal terms of heat flux in generalized heat transport equations have no contribution to the flux-limited behaviors. Another nonlinear effect that will affect the flux-limited behaviors is the temperature-dependent material properties, which may become important in large temperature difference situations and should be explored in future work. The current gray linear approximation made in deriving the phonon hydrodynamic model Eq. (1) could be generalized to include more realistic phonon dispersion relations. The phonon group speed will depend on the frequency, and influences the saturation value of heat flux, which makes the analysis of flux-limited behaviors a challenging task insufficiently explored up to now.

Acknowledgements

Y. Guo and M. Wang acknowledge the financial support of the NSF grant of China (Nos. 51176089, 51321002), the Key Basic Scientific Research Program (2013CB228301) and the Tsinghua University Initiative Scientific Research Program. D. Jou acknowledges the financial support of the Spanish Ministry of Economy and Competitiveness under grant FIS2012-32099, and Ministry of Science and Innovation under grant CSD2010-00044 (Consolider project Nanotherm).

References

[1] D. Jou, J. Casas-Vázquez, G. Lebon, Extended Irreversible Thermodynamics, Springer, Heidelberg, 2010.
[2] G. Chen, Nanoscale Energy Transport and Conversion: A Parallel Treatment of Electrons, Molecules, Phonons, and Photons, Oxford University Press, New York, 2005.
[3] A. Majumdar, Microscale heat conduction in dielectric thin films, J. Heat Transf. 115 (1993) 7–16.
[4] G. Chen, Ballistic-diffusive heat-conduction equations, Phys. Rev. Lett. 86 (2001) 2297–2300.
[5] V.A. Cimmelli, Different thermodynamic theories and different heat conduction laws, J. Non-Equilib. Thermodyn. 34 (2009) 299–333.
[6] G. Lebon, Heat conduction at micro and nanoscales: a review through the prism of extended irreversible thermodynamics, J. Non-Equilib. Thermodyn. 39 (2014) 35–59.
[7] D.G. Cahill, P.V. Braun, G. Chen, D.R. Clarke, S. Fan, K.E. Goodson, P. Keblinski, W.P. King, G.D. Mahan, A. Majumdar, H.J. Maris, S.R. Phillpot, E. Pop, L. Shi, Nanoscale thermal transport. II. 2003–2012, Appl. Phys. Rev. 1 (2014) 011305.
[8] Y. Guo, M. Wang, Phonon hydrodynamics and its applications in nanoscale heat

transport, Phys. Rep. 595 (2015) 1–44.
[9] N. Yang, G. Zhang, B. Li, Violation of Fourier's law and anomalous heat diffusion in silicon nanowires, Nano Today 5 (2010) 85–90.
[10] V.A. Cimmelli, A. Sellitto, D. Jou, Nonlinear evolution and stability of the heat flow in nanosystems: beyond linear phonon hydrodynamics, Phys. Rev. B 82 (2010) 184302.
[11] C. Cattaneo, Sulla conduzione del calore, Atti Semin. Mat. Fis. Univ. Modena 3 (1948) 21.
[12] P. Vernotte, Les paradoxes de la théorie continue de l'équation de la chaleur, C. R. Acad. Sci. 246 (1958) 3154–3155.
[13] D.D. Joseph, L. Preziosi, Heat waves, Rev. Mod. Phys. 61 (1989) 41–73.
[14] W. Dreyer, H. Struchtrup, Heat pulse experiments revisited, Contin. Mech. Thermodyn. 5 (1993) 3–50.
[15] D.Y. Tzou, A unified field approach for heat conduction from macro- to microscales, J. Heat Transf. 117 (1995) 8–16.
[16] D.Y. Tzou, Macro- to Microscale Heat Transfer: The Lagging Behavior, John Wiley & Sons, Ltd., Chichester, 2015.
[17] F.X. Alvarez, D. Jou, A. Sellitto, Phonon hydrodynamics and phonon-boundary scattering in nanosystems, J. Appl. Phys. 105 (2009) 014317.
[18] Z.-Y. Guo, Motion and transfer of thermal mass-thermal mass and thermon gas, J. Eng. Thermophys. 27 (2006) 631–634.
[19] M. Wang, N. Yang, Z.-Y. Guo, Non-Fourier heat conductions in nanomaterials, J. Appl. Phys. 110 (2011) 064310.
[20] M. Wang, Z.-Y. Guo, Understanding of temperature and size dependences of effective thermal conductivity of nanotubes, Phys. Lett. A 374 (2010) 4312–4315.
[21] M. Wang, X. Shan, N. Yang, Understanding length dependences of effective thermal conductivity of nanowires, Phys. Lett. A 376 (2012) 3514–3517.
[22] V.A. Cimmelli, A. Sellitto, D. Jou, Nonlocal effects and second sound in a nonequilibrium steady state, Phys. Rev. B 79 (2009) 014303.
[23] V.A. Cimmelli, A. Sellitto, D. Jou, Nonequilibrium temperatures, heat waves, and nonlinear heat transport equations, Phys. Rev. B 81 (2010) 054301.
[24] P. Ván, T. Fülöp, Universality in heat conduction theory: weakly nonlocal thermodynamics, Ann. Phys. 524 (2012) 470–478.
[25] A. Sellitto, V.A. Cimmelli, D. Jou, Analysis of three nonlinear effects in a continuum approach to heat transport in nanosystems, Physica D 241 (2012) 1344–1350.
[26] Y. Guo, M. Wang, Thermodynamic framework for a generalized heat transport equation, Commun. Appl. Ind. Math. (2015), in press.
[27] J. Schleeh, J. Mateos, I. Íñiguez-de-La-Torre, N. Wadefalk, P. Nilsson, J. Grahn, A. Minnich, Phonon black-body radiation limit for heat dissipation in electronics, Nat. Mater. 14 (2015) 187–192.
[28] A.L. Moore, L. Shi, Emerging challenges and materials for thermal management of electronics, Mater. Today 17 (2014) 163–174.
[29] C. Levermore, G. Pomraning, A flux-limited diffusion theory, Astrophys. J. 248 (1981) 321–334.
[30] A. Anile, V. Romano, Covariant flux-limited diffusion theories, Astrophys. J. 386 (1992) 325–329.
[31] M. Zakari, D. Jou, A generalized Einstein relation for flux-limited diffusion, Physica A: Stat. Mech. Appl. 253 (1998) 205–210.
[32] M. Danielewski, M. Wakihara, Kinetic constraints in diffusion, in: Defect and Diffusion Forum, Trans. Tech. Publ., 2005, pp. 151–156.
[33] X. Shan, M. Wang, On mechanisms of choked gas flows in microchannels, Phys. Lett. A 379 (2015) 2351–2356.
[34] P. Rosenau, Tempered diffusion: a transport process with propagating fronts and inertial delay, Phys. Rev. A 46 (1992) R7371.
[35] M. Zakari, A continued-fraction expansion for flux limiters, Physica A: Stat. Mech. Appl. 240 (1997) 676–684.
[36] D. Jou, M. Zakari, Information theory and heat transport in relativistic gases, J. Phys. A, Math. Gen. 28 (1995) 1585.
[37] M. Zakari, D. Jou, Nonequilibrium Lagrange multipliers and heat-flux saturation, J. Non-Equilib. Thermodyn. 20 (1995) 342–349.
[38] A. Sellitto, V.A. Cimmelli, Flux limiters in radial heat transport in silicon nanolayers, J. Heat Transf. 136 (2014) 071301.
[39] W. Larecki, Symmetric conservative form of low-temperature phonon gas hydrodynamics, Nuovo Cimento D 14 (1992) 141–176.
[40] Z. Banach, W. Larecki, Nine-moment phonon hydrodynamics based on the maximum-entropy closure: one-dimensional flow, J. Phys. A, Math. Gen. 38 (2005) 8781–8802.
[41] Z. Banach, W. Larecki, Nine-moment phonon hydrodynamics based on the modified Grad-type approach: formulation, J. Phys. A, Math. Gen. 37 (2004) 9805–9829.
[42] Z. Banach, W. Larecki, Chapman–Enskog method for a phonon gas with finite heat flux, J. Phys. A, Math. Theor. 41 (2008) 375502.
[43] S.R. De Groot, P. Mazur, Non-Equilibrium Thermodynamics, Dover Publications, New York, 1962.
[44] I. Müller, T. Ruggeri, Rational Extended Thermodynamics, Springer, New York, 1998.
[45] B.-Y. Cao, Z.-Y. Guo, Equation of motion of a phonon gas and non-Fourier heat conduction, J. Appl. Phys. 102 (2007) 053503.
[46] A. Cepellotti, G. Fugallo, L. Paulatto, M. Lazzeri, F. Mauri, N. Marzari, Phonon hydrodynamics in two-dimensional materials, Nat. Commun. 6 (2015) 6400.
[47] S. Lee, D. Broido, K. Esfarjani, G. Chen, Hydrodynamic phonon transport in suspended graphene, Nat. Commun. 6 (2015) 6290.
[48] J.-P.M. Peraud, N.G. Hadjiconstantinou, Efficient simulation of multidimensional phonon transport using energy-based variance-reduced Monte Carlo formulations, Phys. Rev. B 84 (2011) 205331.